

Area Operators in Holographic Quantum Gravity

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Abstract

We argue that the holographic formula relating entanglement entropy and the area of a minimal surface is the key to *define* the area of surfaces in the (emergent) spacetime from the dual theory on the boundary. So we promote the entropy/area relation to operators to define the *area* observable in a holographic formulation of quantum gravity, then we find a suitable geometric representation for the states, and show that the Ryu-Takayanagi proposal is recovered in the approximation of semiclassical gravity. Finally, we discuss this picture in the example of a AdS-Black hole.

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Introduction

The gauge/gravity correspondence suggests that the spacetime geometry, with certain fixed asymptotic behavior, emerges from the degrees of freedom of an ordinary quantum field theory defined on its asymptotic boundary [1]. We can view this field theory as the exact (non-perturbative) description of gravity even at quantum level. This is what we refer to as holographic quantum gravity (HQG), although we only understand features of this paradigm for semiclassical gravity. We really do not know how the basic properties of the classical and quantum geometry can be built up from the gauge theory, but the expectation is that quantum entanglement between subsystems of the dual field theory should be a key ingredient in this mechanism [2, 3]. One of the most fundamental questions related is *which are* suitable observables of the geometry in a consistent holographic formulation of quantum gravity.

In this essay, we focus on this problem and argue that the *area* of spatial surfaces in a emergent quantum geometry can be an observable of the theory of gravity through a holographic definition. This is done by promoting to operators the holographic relation entropy/area, very studied in the the context of gauge/gravity correspondence [4, 5]. We will show that this can have important consequences to describe the states of HQG in a geometric basis, and also observe that those describing a higher dimensional spacetime are highly entangled, in agreement with the arguments of [2] and [3].

Entanglement entropy and holographic area

In a local quantum field theory, where the set of degrees of freedom can be separated in two disjoint subsystems as $\sigma \cup \bar{\sigma}$, and the Hilbert space as $\mathcal{H}_\sigma \otimes \mathcal{H}_{\bar{\sigma}}$, the entanglement entropy is defined as

$$s(\sigma) = -Tr_\sigma (\rho(\sigma) \log \rho(\sigma)) . \quad (1)$$

If the total system is in a pure state $|\psi\rangle$, the reduced density matrix $\rho(\sigma) = Tr_{\bar{\sigma}} |\psi\rangle\langle\psi|$ (normalized to be $Tr_\sigma \rho(\sigma) = 1$) describes the subsystem σ in a mixed state.

In the most familiar example of holography, the AdS/CFT correspondence, the CFT is an ordinary field theory defined on $S^d \times \mathbb{R}$ and its degrees of freedom live on the sphere S^d . The conjecture is that the states of this theory correspond to spacetimes with the same asymptotic behavior as global AdS_{d+2} spacetime².

Let N a spacial slice of an asymptotically AdS spacetime $(M, g_{\mu\nu})$ whose induced (Riemannian) metric is h_{ij} , and $\partial N = S^d$. This corresponds to a fixed time in the CFT.

There is a remarkable holographic formula proposed by Ryu and Takayanagi [4, 5], which has been extensively tested in the last years [6], that relates the entanglement entropy (1) in a strongly coupled quantum field theory and the area of a special surface in the dual holographic spacetime:

$$s(\sigma) = \frac{1}{4l^2} a_\gamma(\Sigma_{min}) \quad , \quad \sigma \subset S^d \quad (2)$$

² $S^d \times \mathbb{R}$ is the asymptotic boundary of the global AdS_{d+2} spacetime.

where $a_\gamma(\Sigma)$ denotes the area of the d -dimensional hypersurfaces $\Sigma(\subset N)$ homolog to σ , such that $\partial\Sigma = \partial\sigma \equiv \gamma$; and Σ_{min} is the one that minimizes this area. The constant $l^2 \equiv G_N^{(d+2)}$ is a fundamental area of the theory of gravity in $d+2$ dimensions, in the context of AdS/CFT it is related to the parameters of the string theory.

This relation should play a essential role in viewing the area as an emergent concept in holographic gravity. So by computing an entropy in the field theory, something that depends only on the state, one in principle might obtain the area of certain surfaces in the dual spacetime and view this as an *observable* of the emergent geometry. Then it would be promoted to operator in some specific formulation of HQG.

Let us notice that the entanglement entropy (1) can be expressed as the expectation value of an operator defined such that

$$\hat{S}(\sigma) = -\log \rho(\sigma) , \quad (3)$$

since

$$s(\sigma) = \langle \psi | \hat{S}(\sigma) | \psi \rangle = -Tr_\sigma \left(\rho(\sigma) \hat{S}(\sigma) \right) = -Tr_\sigma \left(\rho(\sigma) \log \rho(\sigma) \right) . \quad (4)$$

So one can name this the (entanglement) *entropy operator* for the reduced system σ . Since here we assume that $|\psi\rangle$ is the ground state of the whole system, this is what is known in the literature as the modular Hamiltonian [7].

The area operator and HQG states

The natural expectation is that in the quantum gravity regime of AdS/CFT, the space-time metric is fluctuating, and thus it is difficult to give a background independent definition of the area of surfaces. So then, the right hand side of the formula (2) seems to be senseless in a quantum geometry. One can simply view Σ_{min} in (2) as a particular emergent surface in a holographic space whose area is given by the expectation value of the entropy operator, or adopt a less conservative point of view, and admit that this operator may be interpreted as holographic dual to the area observable in a quantum theory of gravity.

Thus, if the area is also promoted to an operator of the dual quantum gravity by means of

$$\hat{S}(\sigma) = \frac{1}{4l^2} A_\gamma . \quad (5)$$

Let us remark that here A_γ stands for an Hermitian operator on the reduced Hilbert space \mathcal{H}_σ , which unlike (2), is not referred to *any special surface* in the space. Let us discuss now the context in which this can make sense.

In the approximation semiclassical of gravity, each state $|\psi\rangle \in \mathcal{H}$ of the conformal field theory on S^d corresponds to a space-like hypersurface (N, h_{ij}) of some specific asymptotically AdS_{d+2} spacetime, such that $\partial N = S^d$. So in the quantum gravity regime, each state

$|\psi\rangle$ should correspond to a specific *initial* wave function of the spatial geometry, formally represented as $\psi_g = \psi_g(N, h_{ij}, \phi)$, where the conventional configuration variables of quantum gravity are N, h_{ij} , and ϕ stands for (bulk) matter fields. For instance, the wave function for the ground state can be defined using a path integral approach, by summing over Euclidean compact spacetimes with boundary conditions (N, h_{ij}) and the same asymptotics as global Euclidean AdS spacetime [8]. This is the Hartle-Hawking construction of the wave function³.

Nevertheless, we will argue here that in HQG, the states can be represented in another suitable basis related to the bulk geometry: a set of d -dimensional spacelike surfaces $\Sigma \subset N$

By virtue of (3) and (5) we have that the reduced density matrix can be expressed in terms of the area operator

$$\rho(\sigma) = C e^{-A_\gamma/4l^2}, \quad (6)$$

where C is the normalizing constant. Remarkably, this is an object entirely referred to the (quantum) theory of gravity.

Naturally, the factorization $\mathcal{H}_\sigma \otimes \mathcal{H}_{\bar{\sigma}}$ implies that the dual \mathcal{H}_{grav} splits in some way such that part of the information on the quantum state of the spacetime was traced out, and the mixed state (6) represents it in a *coarse grained* description of quantum gravity. In fact, (6) is what describes the remaining/effective gravitational degrees of freedom. We will see now that precisely these should be related to a structure of surfaces whose area is quantized.

By virtue of eqs. (3) and (5), A_γ is positive semidefinite, so it can be diagonalized by a orthonormal basis of \mathcal{H}_σ . Its eigenvalues (areas by definition) are attributes of certain d -dimensional spacelike surfaces, and so the spectrum shall be *literally* interpreted as the areas of a family of d -dimensional surfaces $\Sigma \subset N$, homolog to σ and such that $\gamma = \partial\Sigma$. Thus for a given spatial topology N , the area operator can take a definite value for states associated to such surfaces, say

$$A_\gamma |\Sigma_i, \dots\rangle = a(\Sigma_i) |\Sigma_i, \dots\rangle. \quad (7)$$

So A_γ has a spectral decomposition such that $A_\gamma = \sum_i a(\Sigma_i) |(\Sigma, \dots)_i\rangle \langle (\Sigma, \dots)_i|$ where $|(\Sigma, \dots)_i\rangle$ are orthonormal states in \mathcal{H}_σ , and \dots denote the set of quantum numbers that characterize completely each surface i in order to have a complete nondegenerate spectrum of states; for instance, since σ is homolog to Σ_i , one can specify each of these surfaces by giving an embedding field $\mathbf{z}_i : \sigma \mapsto \Sigma_i (\subset N)$, allowing the pullback of extra geometrical structures; in particular, it would provide a $d \times d$ Riemannian metric on each Σ_i . The index i labels a basis of eigenstates of the area operator, and it might be continuum or discrete depending on the structure of the Hilbert space of the dual field theory (eq. (3)). Below we discuss more this point in the example of an AdS black hole.

Near the boundary σ , the manifold N can always be described as a foliation in a one-parameter family of surfaces homolog to σ ; then in this sense, if the quantum spectrum of areas is discrete, one can view it as *approximating* certain foliation of N .

³This approach was applied to the context of AdS/CFT to describe eternal black holes [9]

Then, if $\{(\Sigma_i, \dots)\}$ form a complete basis for the reduced Hilbert space, the HQG general wave functions shall be functionals of these surfaces $\psi = \psi(\Sigma_i, \dots)$ and the mean value of the area operator expresses as,

$$\langle \psi | A_\gamma | \psi \rangle = \sum_i a(\Sigma_i) |\psi(\Sigma_i, \dots)|^2, \quad (8)$$

for an arbitrary state $|\psi\rangle \in \mathcal{H}_\sigma$. This suggests that co-dimension 2 surfaces can constitute configuration variables more convenient for holographic gravity.

By virtue of the formula (6), one can represent the (coarse-grained) *state of the spacetime*, as a probability distribution on geometries labeled by surfaces $\Sigma_i \subset N$:

$$\rho(\sigma) = \sum_i e^{-a(\Sigma_i)/4l^2} |(\Sigma_i, \dots)\rangle \langle (\Sigma_i, \dots)|. \quad (9)$$

A *purification* of this is a state $|\psi_\gamma\rangle \in \mathcal{H}_\sigma \otimes \mathcal{H}_{\bar{\sigma}}$ such that

$$\rho(\sigma) = \text{Tr}_{\bar{\sigma}} (|\psi_\gamma\rangle \langle \psi_\gamma|) \quad (10)$$

Since a similar analysis should be possible for the complement $\bar{\sigma}$, we have that

$$\rho(\bar{\sigma}) = \sum_i e^{-a(\bar{\Sigma}_i)/4l^2} |(\bar{\Sigma}_i, \dots)\rangle \langle (\bar{\Sigma}_i, \dots)|. \quad (11)$$

In particular if we choose $\sigma = \bar{\sigma}$, the respective spectra of surfaces shall be coincident and the pure state writes in geometrical variables as:

$$|\psi_\gamma\rangle = \sum_i e^{-a(\Sigma_i)/8l^2} |(\Sigma_i, \dots)\rangle \otimes |(\bar{\Sigma}_i, \dots)\rangle = \sum_i e^{-a(\bar{\Sigma}_i)/8l^2} |(\Sigma_i, \dots)\rangle \otimes |(\bar{\Sigma}_i, \dots)\rangle \in \mathcal{H}_{grav} \quad (12)$$

is the quantum state of the spacetime written in geometric variables⁴.

This state can literally be interpreted as a *quantum superposition* of (geometric) states characterized by surfaces of area a , or simply as a quantum superposition of surfaces in N .

⁴This expression resembles the general proposal done on Ref. [3]

The minimal surface proposal

Let us now show that our quantization hypothesis (5) is consistent with the Riu-Takayanagi prescription (2). In fact the minimal surface proposal can be derived only by assuming that the entropy/area can be promoted to operators according to the rule (5).

In fact, by taking the expectation value of (5), and using (3) and (5) we obtain

$$\langle \hat{S}(\sigma) \rangle = \frac{1}{4} \langle A_\gamma \rangle = \frac{1}{4l^2} \text{Tr}_\sigma \rho(\sigma) A_\gamma = \frac{C}{4l^2} \text{Tr}_\sigma e^{-A_\gamma/4l^2} A_\gamma \quad (13)$$

where $C^{-1} \equiv \text{Tr}_\sigma e^{-A_\gamma/4l^2}$. If we express it in a basis such that A_γ is diagonal, say $\{|\Sigma_i \cdots > / \partial \Sigma_i = \gamma\}$, results

$$s(\sigma) = \langle \hat{S}(\sigma) \rangle = \frac{C}{4l^2} \sum_i e^{-a(\Sigma_i)/4l^2} a(\Sigma_i) \quad (14)$$

From this expression it can be seen that the dominant term in this sum is the one with the minimum value of the area. So in a standard saddle point approximation for $l^2 \ll a(\Sigma_i)$ ($\forall i$) we recover

$$s(\sigma) \approx \frac{C}{4l^2} e^{-a(\Sigma_{min})/4l^2} a(\Sigma_{min}) \approx \frac{1}{4l^2} a(\Sigma_{min}) , \quad (15)$$

where we have used that $C^{-1} \approx e^{-a(\Sigma_{min})/4l^2}$.

Entanglement, mixed states, and the emergent space

Let us notice the similarity of (12) with the thermal density matrix in a canonical ensemble. So the simplest interpretation of $\rho(\sigma)$ is viewing this as a mixed state with a distribution of probability given in terms of the observable A_γ in HQG. In this way, the $d+1$ -dimensional 'space' N emerges as a distribution of surfaces, in a similar way to a distribution of particles among the states of gas. Thus $e^{-a/4l^2}$ is the Boltzman-like probability, or statistic frequency, of having a surface of area a in the space N . In fact N (foliated in Σ 's) should be thought as an ordinary macroscopic system in a mixed state, which is probed through a large number of independent observations (*ensemble*) in the space-time (see Fig. 1).

Moreover, this result emphasizes that entanglement is fundamentally related to the emergence of the space(time); in fact, let us notice that if the entanglement entropy vanishes ($s(\sigma) = 0$), then the entropy/area operator is totally degenerate (the eigenvalues are 0 or 1) and so there is *only one* emergent surface state $\Sigma \subset N$ ⁵ rather than a family or foliation $\{\Sigma\}_i \approx N$ filling the space. Thus, in the sense discussed here, one would not have a holographic $d+1$ -dimensional emergent space.

We will see below that the cited connection with a canonical ensemble is clear in the context of black holes, where the energy spectrum of CFT can be explicitly related to the spectrum of surfaces in the gravity side.

⁵Or many with coincident areas.

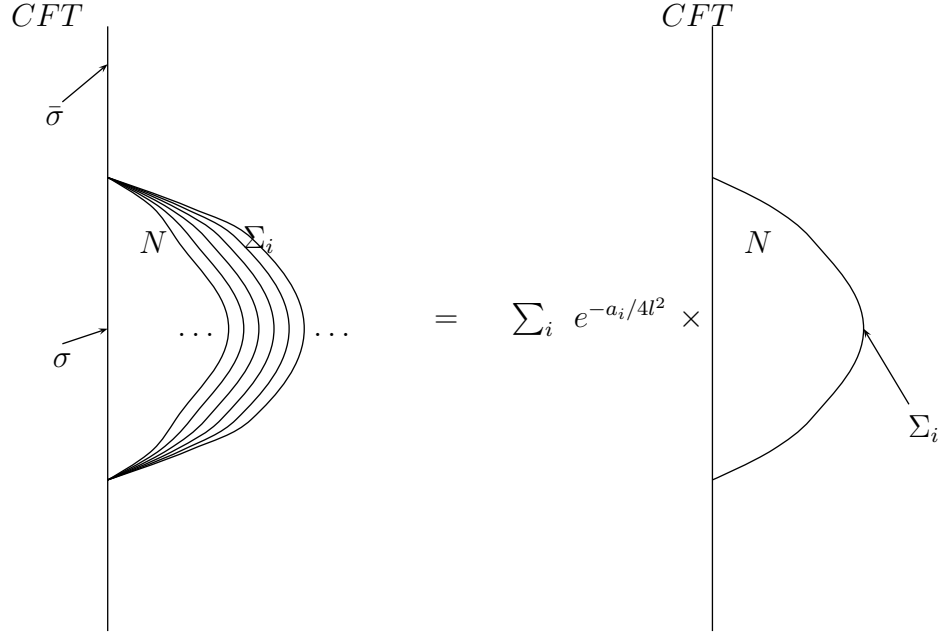


Figure 1: This shows schematically the state (12), that represents the space N as quantum superposition of surface states.

Thermal states and black holes

Thermofield dynamics (TFD) [10, 11] seems to be the most appropriate framework to describe the dual holographic of eternal (AdS) black holes [9], and any spacetime with two asymptotic AdS regions causally disconnected [3], which has thermodynamic properties. This is a well understood context where entanglement play a central role and we have a clear interpretation of what the dual spacetime is [9], so most of the ideas and results discussed previously can be tested and interpreted more clearly.

In the TFD formalism the statistical properties of a system are described by pure states in $\mathcal{H} \otimes \tilde{\mathcal{H}}$, where \mathcal{H} is the Hilbert space of the accessible subsystem and $\tilde{\mathcal{H}}$ is a (decoupled) copy referred to as the *thermofield double*. So the thermal equilibrium corresponds to the entangled state

$$|\psi(\beta)\rangle\rangle = Z^{-1} \sum_n e^{-\beta E_n/2} |n\rangle \otimes |\tilde{n}\rangle \quad (16)$$

where n represents the n th energy eigenstate of the Hamiltonian H , and its copy \tilde{n} spans $\tilde{\mathcal{H}}$. By tracing $|\psi(\beta)\rangle\langle\psi(\beta)|$ over $\tilde{\mathcal{H}}$ one indeed recovers the conventional thermal density operator $\rho(\beta) = Z^{-1} e^{-\beta\hat{H}/2}$.

In TFD, the thermodynamic entropy is regarded as *entanglement entropy* between both sectors, and it is defined as *operator*. For instance, in a free quantum field theory this operator can be expressed *a priori* in terms of the canonical occupation number operator \hat{N}_n of the mode n as [10]:

$$\hat{S} = - \sum_n \left[\hat{N}_n \log \left(\hat{N}_n \right) - \left(1 + \hat{N}_n \right) \log \left(1 + \hat{N}_n \right) \right]. \quad (17)$$

For a thermal distribution \hat{N}_n is a function of β (see refs [10, 11] for details); and in this case, it is not difficult to show that the state (16) can be written in terms of this operator as ⁶

$$|\psi(\beta)\rangle\rangle = e^{-\hat{S}(\beta)/2} \sum_n |n\rangle \otimes |\tilde{n}\rangle. \quad (18)$$

Then, the reduced density matrix results

$$\rho(\beta) \equiv \text{Tr}_{\tilde{\mathcal{H}}} |\psi(\beta)\rangle\rangle \langle\psi(\beta)| = e^{-\hat{S}(\beta)}, \quad (19)$$

which indeed is consistent with the expression (3).

On the other hand, if the system is CFT and $\sigma \equiv S^d$, the complement/TFD-double $\bar{\sigma}$ is another *disconnected* copy of S^d . In this case we know that the state (16) is *literally dual* to the maximally extended AdS-Schwarzschild black hole [9].

Thus, according to the holographic proposal (5) we may write

$$\rho(\beta) = C e^{-A/4l^2} \quad (20)$$

where here A has a spectrum of surfaces Σ_i homolog to S^d , and so by the same arguments followed in the previa sections we obtain that the state can also be expressed as (12). Moreover, if we compute as before the mean value of A for this distribution in the semiclassical gravity approximation, results that the observed entropy of the black hole is

$$s(\beta) \approx a(\Sigma_{min})/4l^2. \quad (21)$$

This is consistent with the Bekenstein-Hawking law remembering that, in a static black hole, the surface Σ_{min} ($\partial\Sigma_{min} = \partial\sigma$) wraps around the horizon as $\sigma \rightarrow S^d$, and so the regularized area coincides with that of the event horizon [13]. In this sense one might observe that there is no surfaces Σ_i ($\sim S^d$) probing the space behind the horizon because $a(\Sigma_i) \geq a_{min}$ [14]

Furthermore, if the CFT Hamiltonian commutes with \hat{S} and the proposal (5) is correct, the expression (20) (or (12)) coincides exactly with $\rho(\beta) = Z^{-1} e^{-\beta\hat{H}/2}$ (or (16)), and thus we

⁶A detailed demonstration of this expression may be found in Refs. [12].

have a remarkable interpretation for each CFT energy eigenstate $|n\rangle$: as *dual to a surface state* $(\Sigma, \dots)_n$ in the gravity side.

In the weak coupling limit of the dual field theory, the state (16) supposedly describes a dual spacetime in the quantum gravity regime, that one might interpret as a quantum black hole. In this case, because of the compactness of S^d , the energy modes n are discrete and by virtue of the formula (17) (and (5)), the entropy/area of the Σ_n 's ($\sim S^d$) are *quantized*.

Conclusions

In this essay, we have defined holographically an area operator involving the entanglement entropy between subsystems of the dual theory in the boundary, and it was done consistently with the holographic recipe [4] in the classical limit.

We have argued that this simple assumption leads to a specific form of the (mixed) state of the spacetime in terms of a geometrical representation related to co-dimension two surfaces. This might be a suitable configuration basis for the Hilbert space of HQG that shall be more investigated.

It is fascinating that this pictures manifestly captures the significance of the quantum entanglement in the holographic emergence of the spacetime, in agreement with arguments previously presented [2] [3].

Let us mention finally that the idea of quantizing the area has also been developed in the context of *loop quantum gravity*, but the spirit and construction presented here are completely different.

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